Math 201-Fall-2015-16

Calculus and Analytic Geometry III, sections 25,26,27,28

Quiz 1, October 1, 2015-Duration: 60 minutes

YOUR NAME: Key 1

YOUR AUB ID #:

INSTRUCTIONS: Closed book and notes. **NO CACULATORS ALLOWED**. Turn **OFF** and put away any cell phones.

GRADES:

1	2	3	4	5			

1. (20 pts.) Find the following limits

10 pts (a)

$$\lim_{n\to\infty}\left(\frac{2n+5}{2n-1}\right)^{3n}$$

5 pts for doing correctly ... any of limits involving exp

$$\lim_{n\to\infty}\left(\frac{2n+5}{2n-1}\right)^{3n} = \lim_{n\to\infty}\left(\frac{1+\frac{5}{2n}}{1-\frac{1}{2n}}\right)^{2n\cdot\frac{3}{2}}$$

$$= \lim\left(\frac{\left(1+\frac{5}{2n}\right)^{2n}}{\left(1-\frac{1}{2n}\right)^{2n}}\right)^{3/2} = \left(\frac{e^{5/2}}{e^{-1/2}}\right)^{3/2} = (e^{3})^{3/2} = e^{9/2}.$$

10 pts (b)

$$\lim_{n\to\infty}\frac{1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\cdots+\frac{1}{\sqrt{n}}}{6\sqrt{n}}$$

2 pts

$$\int_1^{n+1}\frac{1}{\sqrt{x}}\,dx < 1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}} < 1+\int_1^{n}\frac{1}{\sqrt{x}}\,dx$$

$$2\sqrt{x}\Big|_1^{n+1} < \qquad < 1+2\sqrt{x}\Big|_1^{n}$$

4 pts

$$2\sqrt{n+1}-2 < 1+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}} < 2\sqrt{n}-1$$

Divide by $6\sqrt{n}$

2 pts

$$\frac{\sqrt{n+1}-1}{3\sqrt{n}} < \frac{1+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}}{6\sqrt{n}} < \frac{2\sqrt{n}-1}{6\sqrt{n}}$$

Take limits as $n\to\infty$, the result is that

2 pts

$$\lim_{n\to\infty}\frac{1+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}}{6\sqrt{n}} = \frac{1}{3}.$$

2.

(a) (15 pts.) Test each of the following series for convergence

$$\sum_{n=1}^{\infty} \frac{n^2+1}{3^n}, \quad \sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

1st: By ratio test: $\lim \frac{(n+1)^2+1/3^{n+1}}{(n^2+1)/3^n} = \lim \frac{1}{3} \cdot \frac{(n+1)^2+1}{n^2+1}$

7 pts $= \lim \frac{1}{3} \cdot \frac{n^2+2n+2}{n^2+1} = \lim \frac{1}{3} \cdot \frac{1+\frac{2}{n}+\frac{2}{n^2}}{1+\frac{1}{n^2}} = \frac{1}{3}$.

$\therefore$ series converges

2nd Integral test: $\int_2^{\infty} \frac{1}{x \ln x} dx = \lim_{A \to \infty} \int_2^A \frac{1}{x \ln x} dx$

7 pts. $= \lim_{A \to \infty} \left(\ln \ln x \Big|_2^A \right) = \lim_{A \to \infty} (\ln \ln A - \ln \ln 2) = \infty$

$\therefore$ series diverges.

1 pt. Note: $\frac{1}{x \ln x}$ is decreasing because $x \ln x$ is increasing

(c) (10 pts.) Find all values of p for which the following series converges

$$\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \sin \frac{1}{\sqrt{n}} \right)^p$$

We use L.C.T. to compare this series with

3 $\sum_{n=1}^{\infty} \left(\frac{1}{(\sqrt{n})^3} \right)^p$.

2 pts

$\lim_{n \to \infty} \left(\frac{\frac{1}{\sqrt{n}} - \sin \frac{1}{\sqrt{n}}}{\left(\frac{1}{\sqrt{n}}\right)^3} \right)^p = \lim_{x \to 0} \left(\frac{x - \sin x}{x^3} \right)^p$

6 3 $= \lim_{x \to 0} \left(\frac{x - (x - x^3/3! + x^5/5! - \dots)}{x^3} \right)^p = \lim_{x \to 0} \left(\frac{\frac{x^3}{6} - \frac{x^5}{5!} + \dots}{x^3} \right)^p$

$= \left(\frac{1}{6} \right)^p$.

$\therefore$ The two series behave alike.

4 pts. But $\sum_1^{\infty} \frac{1}{n^{\frac{3}{2}p}}$ converges iff $\frac{3}{2}p > 1$, iff $p > \frac{2}{3}$.

$\therefore$ the given series converges for $p > 2/3$.

Root test 3 pts

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3. (15 pts.) Find all values of x for which the following series is convergent.

$$\sum_{n=0}^{\infty} \frac{4^n (n!)^2 (x-1)^n}{(2n)!}$$

abs 1

Applying the ratio test :

5pt

$$\rho = \lim_{n\to\infty} \left| \frac{4^{n+1}((n+1)!)^2(x-1)^{n+1}}{(2n+2)!} \Big/ \frac{4^n (n!)^2 (x-1)^n}{(2n)!} \right|$$

$$⑤ = \lim_{n\to\infty} |(x-1)| \cdot \frac{4.(n+1)^2}{(2n+1)(2n+2)} = \frac{4}{4}|x-1| = |x-1|$$

∴ series converges absolutely for each x with

② $|x-1| < 1$, $|x-1| < 1$, $-1 < x-1 < 1$

3 pts

$0 < x < 2$. ②

End points.

If $x = 2$: $\sum_{n=0}^{\infty} \frac{4^n (n!)^2}{(2n)!}$

we notice that $\frac{4^{n+1}((n+1)!)^2}{(2n+2)!} \div \frac{4^n (n!)^2}{(2n)!} = \frac{4(n+1)^2}{(2n+1)(2n+2)}$

4 pts

$$= \frac{4n^2 + 8n + 4}{4n^2 + 6n + 2} > 1.$$

∴ The terms increase in value. The limit of the n^{th} term is not zero, the series diverges.

If $x = 0$, $\sum_{0}^{\infty} \frac{(-1)^n . 4^n (n!)^2}{(2n)!}$

again the terms increase in absolute value so n^{th} term does not tend to zero, series diverges.

Hence series converges Absolutely for $0 < x < 2$ and otherwise diverges

4. (a) (~~15~~ 17 pts.) Find the Taylor series expansion of $f(x) = \frac{(x-3)^4}{x+5}$ about the point $a = 3$, and use it to find $f^{(n)}(3)$. (Hint: First find the Taylor series of $\frac{1}{x+5}$ about $a = 3$).

$$\frac{1}{x+5} = \frac{1}{x-3+8} = \frac{1}{8(1+\frac{x-3}{8})} = \frac{1}{8}\sum_{n=0}^{\infty}\left(-\frac{(x-3)}{8}\right)^n$$ 4

10 pts

$$= \frac{1}{8}\sum_{0}^{\infty}\frac{(-1)^n(x-3)^n}{8^n} = \sum_{n=0}^{\infty}\frac{(-1)^n(x-3)^n}{8^{n+1}}$$

provided $\left|\frac{x-3}{8}\right| < 1$, $|x-3| < 8$. 1

3 pts

$$\therefore f(x) = \frac{(x-3)^4}{(x+5)} = \sum_{n=0}^{\infty}\frac{(-1)^n(x-3)^{n+4}}{8^{n+1}} = \frac{(x-3)^4}{8} - \dots$$

$$= \sum_{n=4}^{\infty}\frac{(-1)^n}{8^{n-3}}(x-3)^n \quad \text{so } \frac{f^{(n)}(3)}{n!} = \text{coeff of } (x-3)^n = \frac{(-1)^n}{8^{n-3}}$$

$n \geq 0$ so $f^{(n)}(3) = \frac{(-1)^n n!}{8^{n-3}}$

$$\frac{f^{(n+4)}(3)}{(n+4)!} = \text{coefficient of } (x-3)^{n+4} = \frac{(-1)^n}{8^{n+1}}.$$

4 pts

$$\therefore \frac{f^{(n)}(3)}{n!} = \frac{(-1)^{n-4}}{8^{n-4+1}} = \frac{(-1)^n}{8^{n-3}} \quad \text{or} \quad f^{(n)}(3) = \frac{(-1)^n n!}{8^{n-3}} \text{ for } n \geq 4$$

otherwise $f^{(n)}(3) = 0$, $n = 0, 1, 2, 3$.

(b) (~~5~~ 3 pts.) If f is the same function as in part (a) above, find the Taylor series of the derivative $f'(x)$ about $a = 3$.

3 pts

We just differentiate the series:

$$f'(x) = \sum_{n=0}^{\infty}\frac{(-1)^n(n+4)(x-3)^{n+3}}{8^{n+1}}.$$

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5. (~~20~~ pts.) Let $\sum_{n=1}^{\infty} a_n$ be an infinite series of positive terms, and S_n its sequence of partial sums. Suppose that we are given that $\lim_{n\to\infty} \frac{a_n}{S_n} = \frac{1}{3}$.

show S_n's ⊕ and increasing so $S_n \not\to 0$

(a) Does the series $\sum_{n=1}^{\infty} a_n$ converge or diverge? Justify your answer.

Suppose $\sum_1^{\infty} a_n$ converges. Then $\lim_{n\to\infty} a_n = 0$ & $\lim_{n\to\infty} S_n = S > 0$

Then $\lim_{n\to\infty} \frac{a_n}{S_n} = (\lim a_n) \lim \frac{1}{S_n} = 0 \cdot \frac{1}{S} = 0$ contrary

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to hypothesis.

Hence the series diverges.

(b) Is $\{a_n\}$ a bounded sequence? Justify your answer.

Since series diverges, $\lim_{n\to\infty} S_n = \infty$. If a_n were bounded,

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we would have $0 < \frac{a_n}{S_n} < \frac{M}{S_n}$ & since $\lim_{n\to\infty} \frac{M}{S_n} = 0$ we

would get $\lim_{n\to\infty} \frac{a_n}{S_n} = 0$ contrary to hypothesis. ∴ a_n is not bounded.

(c) Find $\lim_{n\to\infty} a_n \sin(\frac{1}{3a_n})$, ~~if in addition a_n is increasing~~.

Since $\lim \frac{a_n}{S_n} = 1/3 \Rightarrow \frac{a_n}{S_n} > \frac{1}{4}$ $n > N$, so $\lim a_n = \infty$ since $\lim S_n = \infty$

∴ $\lim \frac{1}{a_n} = 0$.

or $a_n = \left(\frac{a_n}{S_n}\right) S_n$, $\frac{1}{3}$, ∞

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∴ $\lim_{n\to\infty} a_n \sin\frac{1}{3a_n} = \lim_{n\to\infty} \frac{\sin(\frac{1}{3a_n})}{3 \cdot \frac{1}{3a_n}} = \frac{1}{3}$

$a_n > \frac{1}{4} S_n$ but $S_n \to \infty$ so $a_n \to \infty$

(d) Give an example of an infinite series satisfying the condition in part (a).

Take the series $\sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n$.

Then $S_n = \sum_{k=0}^{n} \left(\frac{3}{2}\right)^k = 1 + \frac{3}{2} + \cdots + \left(\frac{3}{2}\right)^n = \frac{1-(3/2)^{n+1}}{1-(3/2)} = \frac{(3/2)^{n+1}-1}{3/2}$

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and $a_n = \left(\frac{3}{2}\right)^n$.

Then $\lim_{n\to\infty} \frac{a_n}{S_n} = \lim \frac{(3/2)^n}{\frac{(3/2)^{n+1}-1}{3/2-1}} = \frac{1}{\frac{3}{2}} \cdot \frac{1}{2} = \frac{1}{3}$.